Feb. 14, 2014

Issues in Curve Sketching

Plan: Look at f vs. f'
- inc/dec
- max/min
- Critical pts

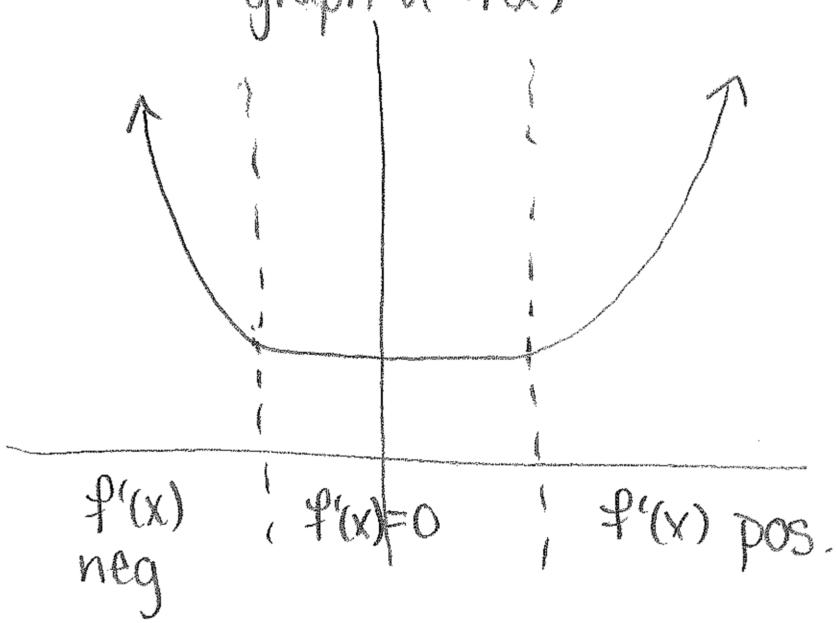
Look at f vs. f''
- Concave up/down
- inflection pts

Recall: f inc $\Rightarrow$ f' pos.

f dec $\Rightarrow$ f' neg

f constant $\Rightarrow$ f' zero

graph of $f(x)$



Finding "Extreme" Values

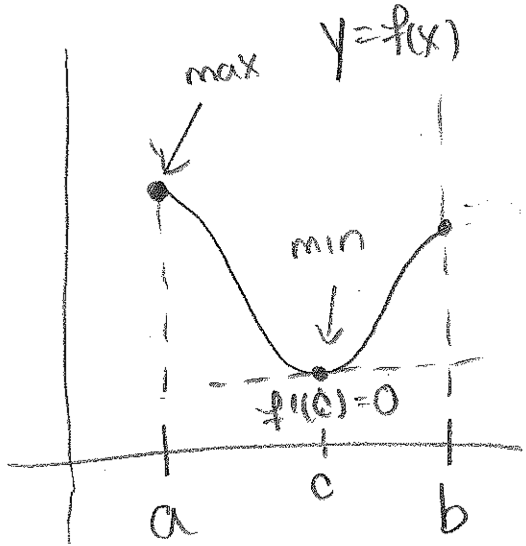
Recall: If we have f continuous on $[a,b]$, then there is a pt in $[a,b]$ where f is maximized and a pt where f is minimized

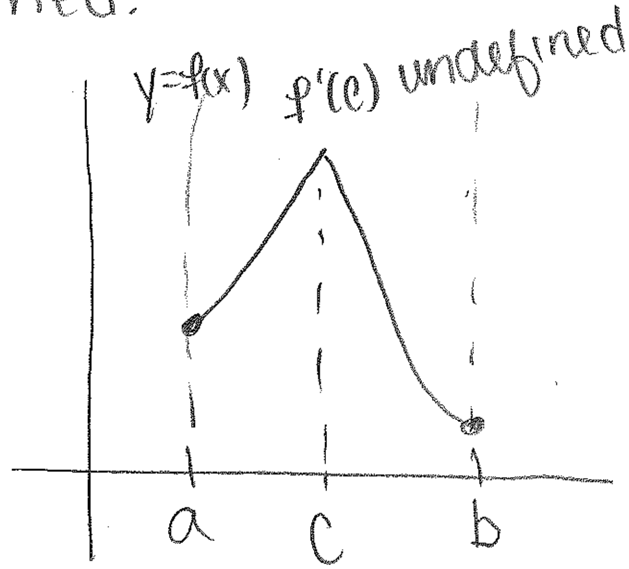
The max/mins will occur at either:

1. an endpoint ($x=a$ or $x=b$)

— OR —

2. at a point c where $f'(c)=0$ OR $f'(c)$ is undefined.





DEF: A <u>critical point</u> of a function f is a number c in the domain of f such that either $f'(c)=0$ OR $f'(c)$ does not exist.

* by the "Recall" above, critical points are the only place a function can change directions.

There are many different types of max/min values we could be interested in:

Sometimes we look at entire domain instead of just [a,b]

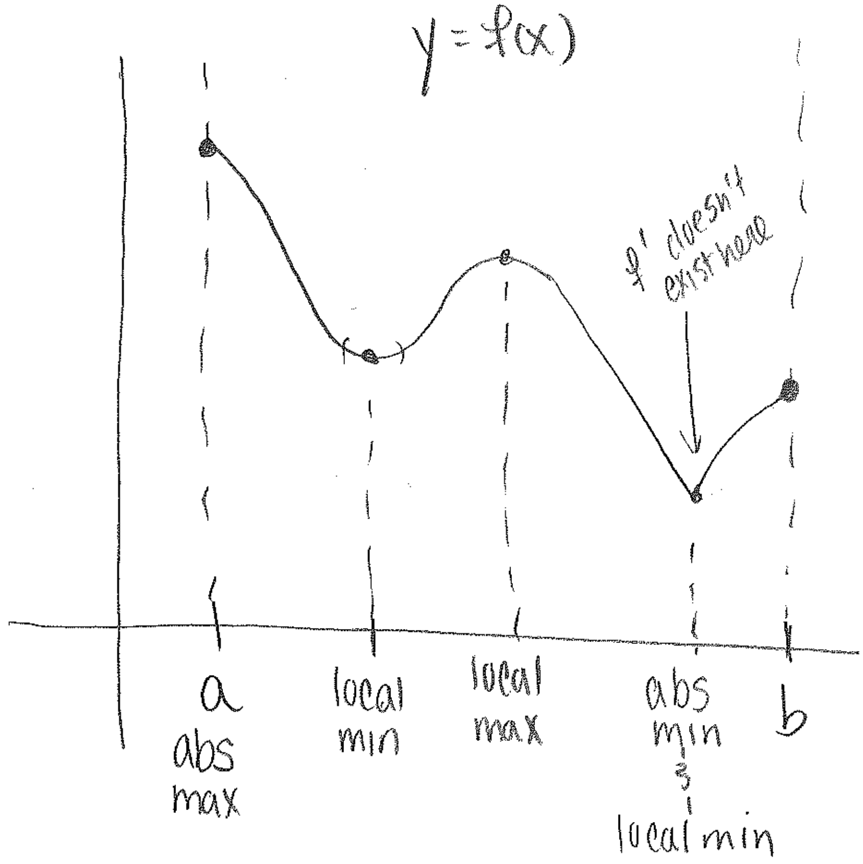
Absolute maximum on $\underbrace{[a,b]}_{\text{some int.}}$: a value c s.t. $f(c) \geq f(x)$ where x is any value in $[a,b]$.

Absolute minimum on [a,b]: A value c s.t. $f(c) \leq f(x)$ for all x in $[a,b]$

Local max: Some c where $f(c) \geq f(x)$ for all values of x "near" c.

Local min: Some c where $f(c) \leq f(x)$ for all values of x "near" c.

* "near" means you can look as close to c as you want.



How to find local min/max:

$\boxed{\text{If } c \text{ is a local min/max of } f \text{ then } c \text{ is a critical pt}}$

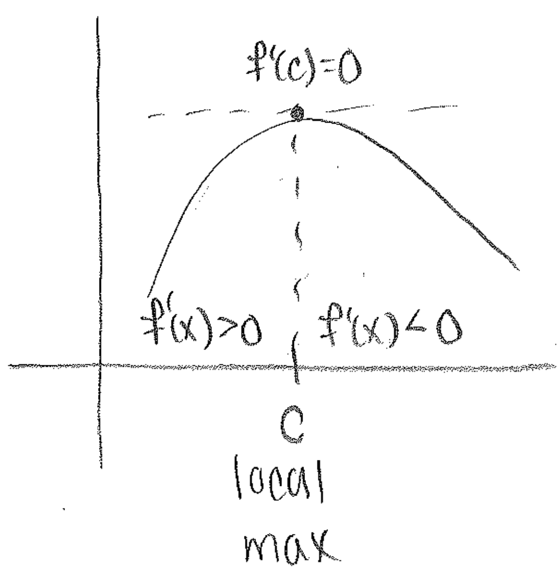
CAUTION: c critical pt does not mean that c is a local min/max.

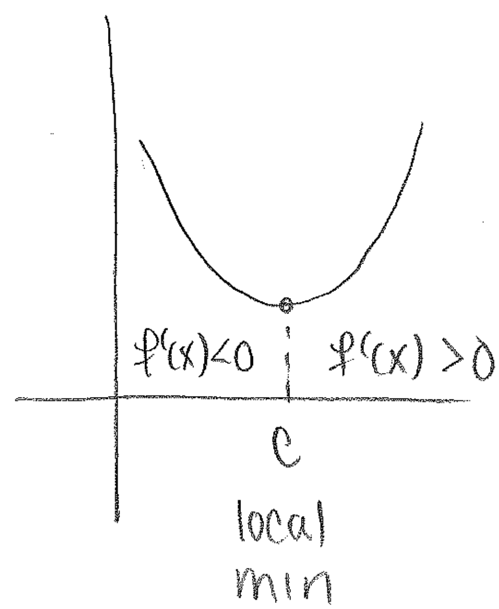
Local min/max test. Say you have a critical point c:

(1) If $f'(x)$ changes from pos. to neg. at c, then f has a local max at c.

(2) If $f'(x)$ changes from neg to pos at c then f has a local min at c.

(3) If $f'(x)$ doesn't change sign at c, nothing special happens





Example: Find local min/max of

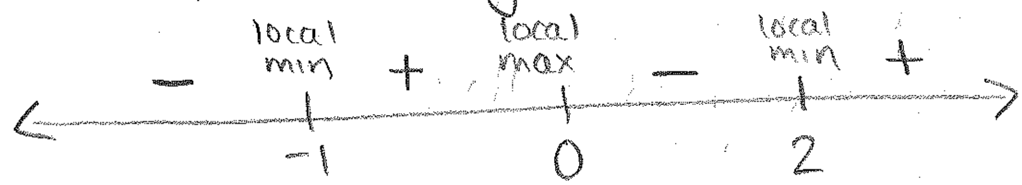
$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2)$
$= 12x(x-2)(x+1)$

Find critical points:
$f'(x)$ is defined everywhere,
so only need to find c where $f'(c) = 0$.

$0 = 12x(x-2)(x+1) \Rightarrow x = 0, -1, 2$

Check pos and neg for derivative:



(this is this same process as for increasing/decreasing)

local min: $-1, 2$
local max: 0

Example: Find local min/max of $f(x) = |x+1|$

$f'(x) = \begin{cases} -1 & x < -1 \\ 1 & x > -1 \end{cases}$ $f'(x)$ never equals zero, but doesn't exist at $x = -1$. So -1 is a critical point.

Check

-1 is a local min

Finding absolute min/max on an interval $[a,b]$

* we are find the one pt where f is largest
and the one pt where f is smallest

(1) Find all critical pts
(remember, these include where f' doesn't exist!)

(2) check values for f at critical pts and endpts. (a and b)
Largest is abs. max
Smallest is abs. min

Ex: Find absolute max/min for
$x^3 - 3x^2 - 9x + 2$ on interval $[-2, 2]$.

$f'(x) = 3x^2 - 6x - 9 = (3x-9)(x+1)$

critical pts: $x = -1, 3$ not in interval
(Note that f' exists everywhere)
so no singular pts

endpts: $-2, 2$

Check $f(x)$: $f(-2) = (-2)^3 - 3(-2)^2 - 9(-2) + 2 = 0$
$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 2 = 7$ largest
$f(2) = 2^3 - 3\cdot 2^2 - 9\cdot 2 + 2 = -20$ smallest

Absolute Max is $\cancel{7}$ 7 (when $x = -1$)

Absolute Min is -20 (when $x = 2$).

Using derivatives to graph functions

Want to graph: $f(x) = \dfrac{x^4+1}{x^2}$

(1) Domain: $(-\infty,0)\cup(0,\infty)$

(2) Increasing/Decreasing

$$f'(x) = \frac{4x^3 \cdot x^2 - (x^4+1)2x}{x^4} = \frac{4x^5 - 2x^5 - 2x}{x^4}$$

$$= \frac{2x(x^4-1)}{x^4} \quad \frac{\cancel{2x}(x^2+1)(x+1)(x-1)}{x^{\cancel{4}3}}$$

$$= \frac{2(x^2+1)(x+1)(x-1)}{x^3}$$

Critical Points:

$f'(x)=0$ when $x=-1, 1$

$f'(x)$ doesn't exist when $x=0$

$-$		$+$		$-$		$+$
	-1		0		1	
dec		inc		dec		inc
	local min		Not local max — 0 not in domain of $f(x)$		local min	

(3) Concavity: $f''(x) = \dfrac{2((4x^3)\cdot x^3 - 3x^2(x^4-1))}{3x^2}$ $\left(f'(x) = \dfrac{2(x^4-1)}{x^3}\right)$

$$= 2\,\frac{x^4+3}{x^4} = 2\left(\frac{x^4}{x^4} + \frac{3}{x^4}\right)$$

$$= 2 + \frac{6}{x^4}$$ always positive

$f(x)$ is always concave up.

Other important things:

(4) Asymptotes?

Horizontal: $\lim_{x \to -\infty} \frac{x^4+1}{x^2} = \infty$ $\lim_{x \to \infty} \frac{x^4+1}{x^2} = \infty$

Vertical: $\lim_{x \to 0^-} \frac{x^4+1}{x^2} = \infty$ $\lim_{x \to 0^+} \frac{x^4+1}{x^2} = \infty$

No horizontal asymptotes

Yes vertical asymp. at $x = 0$

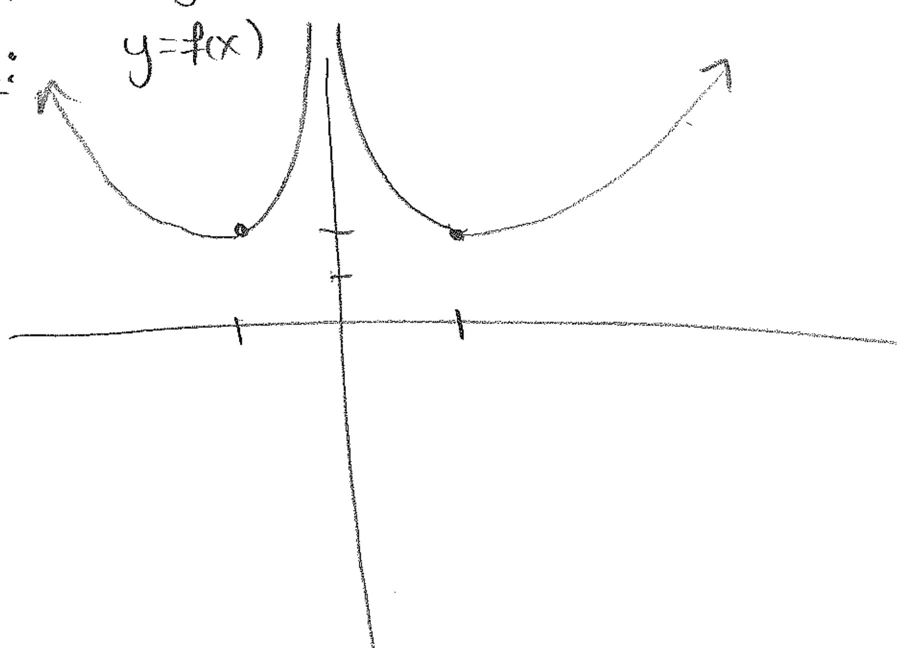
(5) find some landmarks:

- roots of $f(x)$: None
- Calculate $f(x)$ at $x=0$, critical pts and inflection pts.

$f(0) \to \infty$

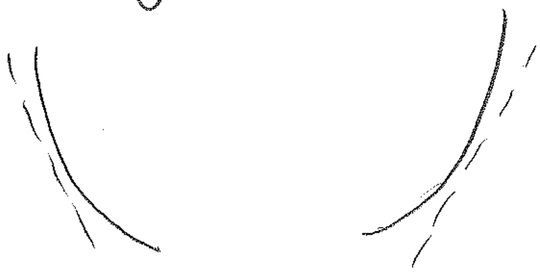
$\left.\begin{array}{l} f(-1) = 2 \\ f(1) = 2 \end{array}\right\}$ local min

Sketch:



Concavity & Second Derivative

Concavity: which way the line is curving.

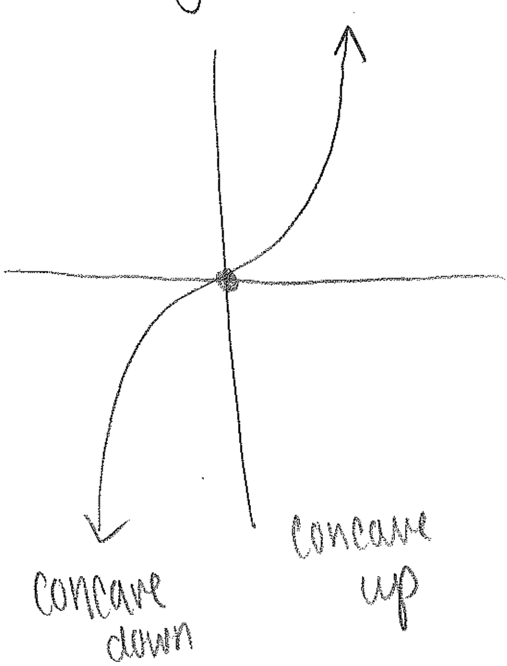


concave up

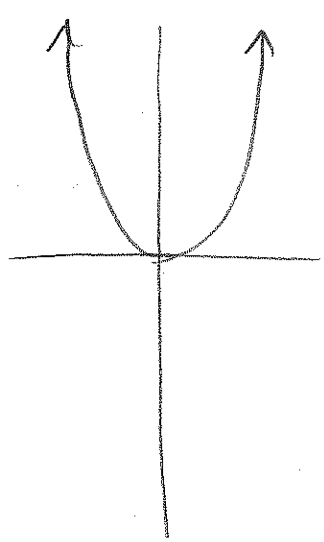
concave down

Second Derivatives: when f'' is pos., f is concave up
when f'' is neg., f is concave down

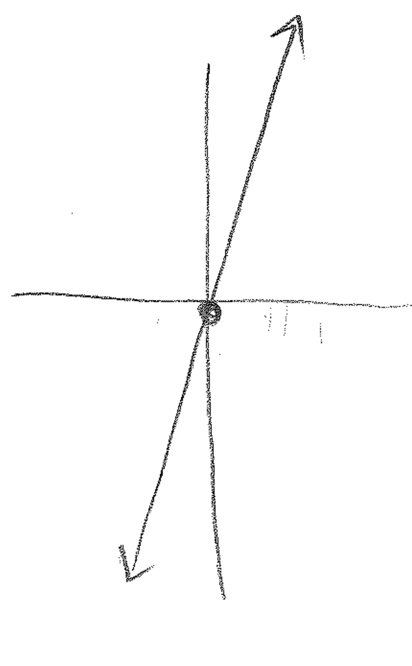
$y = x^3$



$y' = 3x^2$



$y'' = 6x$



Inflection pt: where a graph changes concavity.

* If f has a inflection pt and f'' exists at that pt, the f'' = 0 there. (but only if f'' switches from pos. to neg.)

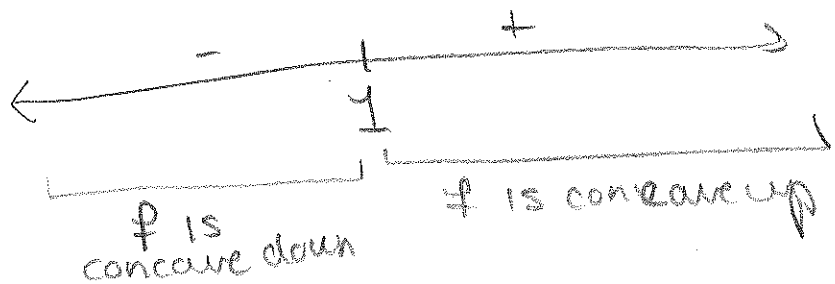
Ex: Find where $f(x) = x^3 - 3x^2 - 9x + 2$ is concave up and down and find any inflection points.

$f'(x) = 3x^2 - 6x - 9$

$f''(x) = 6x - 6$

set $f''(x) = 0$: $0 = 6x - 6$

$x = 1$



f is concave up on $(1, \infty)$

f is concave down on $(-\infty, 1)$

Inflection pt at $x = 1$

Max/Min w/ 2nd Derivative: S'pose f'' is cont. near c

1) If $f'(c) = 0$ and $f''(c) > 0$, f has a local min at c

2) If $f'(c) = 0$ and $f''(c) < 0$, f has a local max at c.